Entanglement in Time

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February 28, 2023

Entanglement in time can be viewed as the (quantum) mutual information between a state $|\psi\rangle$ and a time-evolved state $|\psi\rangle = U(t)\,|\psi\rangle$, and has to do the quantum chaos and thermalization [1, 2]. The time evolution operator $U(t) = e^{-iHt}$ acting on n qubits is a unitary operator described by a $2^n \times 2^n$ dimensional matrix

$$U(t) = \sum_{i,j=1}^{2^n} u_{ji}\,|j\rangle\,\langle i| \tag{1}$$

which can be perceived as a unitary gate that takes in the *Past* state $|i\rangle_{\mathrm{p}}$ and spits out the *Future* state $|j\rangle_{\mathrm{f}}$ with an amplitude u_{ji}, and we usually choose to think of it in terms of a tensor with n input and n output legs. However, it is also natural to map this to a $2n$-qubit state by treating the input and output legs on equal footing

$$|U(t)\rangle = \frac{1}{2^{n/2}} \sum_{i,j=1}^{2^n} u_{ji}\,|i\rangle_{\mathrm{p}} \otimes |j\rangle_{\mathrm{f}} \tag{2}$$

where it is readily to check that the prefactor $\frac{1}{2^{n/2}}$ normalizes the state. Clearly the $|U(t)\rangle$ defined above, which encodes all u_{ji}, is isomorphic to the unitary matrix $U(t)$.

At $t = 0$, the unitary operator is just the identity operator i.e. $U(t = 0) = I$, and Eq. 2 reduces to

$$|I\rangle = |U(t=0)\rangle = \frac{1}{2^{n/2}} \sum_{i,j=1}^{2^n} \delta_{ij}\,|i\rangle_{\mathrm{p}} \otimes |j\rangle_{\mathrm{f}} = \frac{1}{2^{n/2}} \sum_{i}^{2^n} |i\rangle_{\mathrm{p}} \otimes |i\rangle_{\mathrm{f}} \tag{3}$$

which obviously corresponds to

$$I = \sum_{i}^{2^n} |i\rangle\,\langle i| \quad \sim \quad |I\rangle = \frac{1}{2^{n/2}} \sum_{i}^{2^n} |i\rangle_{\mathrm{p}} \otimes |i\rangle_{\mathrm{f}} \tag{4}$$

Note that $|I\rangle$ is a maximally entangled state between two sets of n qubits, i.e. $2n$ qubits consisting of n EPR pairs.

Furthermore, for finite t, we can rewrite $|U(t)\rangle$ in terms of

$$|U(t)\rangle = I_{\mathrm{p}} \otimes U_{\mathrm{f}}(t)\,|I\rangle \tag{5}$$

with $U(t)$ acting on the output legs. This can be readily proved by exploiting the completeness of states, and the change of dummy indices by kronecker delta:

Proof of Eq. 5.

$$
\begin{aligned}
|U(t)\rangle &= \frac{1}{2^{n/2}} \sum_{i,j=1}^{2^n} u_{ji} \, |i\rangle_{\mathrm{p}} \otimes |j\rangle_{\mathrm{f}} \\
&= \frac{1}{2^{n/2}} \sum_{i,j=1}^{2^n} |i\rangle_{\mathrm{p}} \otimes |j\rangle_{\mathrm{f}} \sum_{i',j'} u_{j'i'} \; {}_{\mathrm{f}}\langle i|i'\rangle_{\mathrm{f}} \; {}_{\mathrm{f}}\langle j|j'\rangle_{\mathrm{f}} \\
&= \frac{1}{2^{n/2}} \sum_{ij} |i\rangle_{\mathrm{p}} \otimes |j\rangle_{\mathrm{f}} \; {}_{\mathrm{f}}\langle j| \sum_{i',j'} u_{j'i'} \, |j'\rangle_{\mathrm{f}} \; {}_{\mathrm{f}}\langle i'|i\rangle_{\mathrm{f}} \\
&= \frac{1}{2^{n/2}} \sum_{i} |i\rangle_{\mathrm{p}} \otimes \underbrace{\left(\sum_{j} |j\rangle_{\mathrm{f}} \; {}_{\mathrm{f}}\langle j| \right)}_{=I_{\mathrm{f}}} \underbrace{\left(\sum_{i',j'} u_{j'i'} \, |j'\rangle_{\mathrm{f}} \; {}_{\mathrm{f}}\langle i'| \right)}_{U_{\mathrm{f}}(t)} |i\rangle_{\mathrm{f}} \\
&= \frac{1}{2^{n/2}} \sum_{i} |i\rangle_{\mathrm{p}} \otimes U_{\mathrm{f}}(t) \, |i\rangle_{\mathrm{f}} \\
&= I_{\mathrm{p}} \otimes U_{\mathrm{f}} \, |I\rangle
\end{aligned}
\tag{6}
$$

where in the last step we used Eq. 3. $\square$

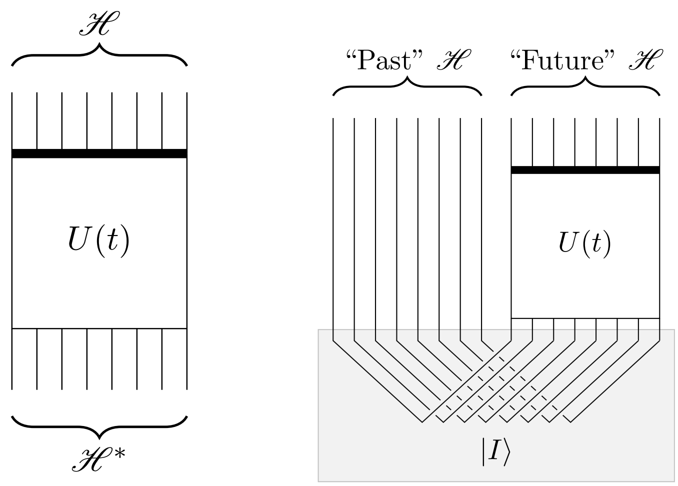

Figure 1: Pictorial representation and explicit construction of the mapping from time evolution operator $U(t)$ to $|U(t)\rangle$ [2]

This gives the following interpretation to the state $|U(t)\rangle$: Starting with a maximally entangled state that is created between a reference system caller "past" and a system called "future", the operator $U(t)$ acts on the "future" part. The resulting state of the full extended Hilbert space is state $|U(t)\rangle$. This is pictorially illustrated in the Fig. 1. By bringing the past and future under the same footing, we can define the density matrix and entanglement entropy in the same way used in the spacial entanglement.

References

[1] Hosur, P., Qi, X.-L., Roberts, D. A. & Yoshida, B. Chaos in quantum channels. *Journal of High Energy Physics* **2016**, 4 (2016). URL https://doi.org/10.1007/JHEP02(2016)004.

[2] Lensky, Y. D. & Qi, X.-L. Chaos and high temperature pure state thermalization. *Journal of High Energy Physics* **2019**, 25 (2019). URL https://doi.org/10.1007/JHEP06(2019)025.