Noether Theorem

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1 Noether theorem in classical mechanics

1.1 Symmetry

In classical mechanics, the action $S[q^i(t)]$ is defined as the integral of lagrangian:

$$S[q^i(t)] = \int dt\ L(q^i, \dot{q}^i, t) \tag{1.1}$$

The crucial concept exploited by Noether is that of an action symmetry, for example, define an action:

$$S[x(t), y(t), z(t)] = \int dt\ \left((\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - H_0 z\right) \tag{1.2}$$

with H_0 a constant. This action is clearly invariant under constant translations in $x(t)$ and $y(t)$, i.e.

$$S[x(t) + x_0, y(t) + y_0, z(t)] = S[x(t), y(t), z(t)] \tag{1.3}$$

The important aspect of this equation is that is holds for **all trajectories** $x(t), y(t), z(t)$. Note t here should be viewed as a dummy index like a coordinate frame, and $x(t), \cdots$ as a field or configuration defined on that frame.

There can be more complicated forms of symmetry. For example, let q^i be a set of generalized coordinates. For an action $S[q^i(t)]$, a set of functions $f^i(t)$ is a symmetry if $S[q^i(t)+f^i(t)] = S[q^i(t)]$ for all $q^i(t)$. In other words, Symmetries are directions in the space spanned by the q_i's on which the action does not change.

For Noether's theorem one is interested in **infinitesimal symmetries**. The functions $f^i(t)$ in the example above will be denoted as $f^i(t) \equiv \delta_s q^i(t)$ (s for symmetry), and $S[q^i(t) + \delta_s q^i(t)]$ is spanned to first order. It is important to note that $q^i(t)$ **and** $\delta q^i(t)$ **are totally independent functions**.

So far we have mentioned the strong version of a symmetry where the action is strictly invariant. **Noether's theorem accepts a weaker version**. This weaker version of symmetry is defined as a function $\delta_s q^i(t)$ such that, for any $q^i(t)$ (meaning for any i and any t-dependent trajectory), the action is **invariant up to a boundary term** K:

$$\delta S[q^i(t), \delta_s q^i(t)] \equiv S[q^i(t) + \delta_s q^i(t)] - S[q^i(t)] = \int \frac{dK}{dt} dt \tag{1.4}$$

which is a function of **both** the configuration $q^i(t)$ and the symmetry $\delta_s q^i(t)$. By analogy, this can be simply perceived as defining a variation as $f(x + dx) - f(x) = f'(x)dx \equiv df(x, dx)$ and relabel d by δ.

Example 1.1 (Rotational symmetry)**.** The action of a point particle that feels a central force is:

$$S[\vec{r}(t)] = \int dt \left(\frac{m}{2}\dot{\vec{r}}^2(t) - V(r)\right) \tag{1.5}$$

It's readily to see the ratational invariance since $\vec{r}^{\mathrm{T}} R^{\mathrm{T}} R \vec{r} = \vec{r}^2$. So it is also true for an infinifesimal rotation $\vec{\alpha}$. For small angle this can be written as

$$\vec{r}'(t) = \vec{r}(t) + \vec{\alpha} \times \vec{r}(t) \tag{1.6}$$

hence

$$\delta_s \vec{r}(t) = \vec{\alpha} \times \vec{r} \tag{1.7}$$

with direction of α penpendicular to the manifold spanned by $\vec{r}$ (e.g. for xy plane, α point into z direction. See appendix for details). Now let us work out what is the boundary term of δS. $V(r)$ only depend on length of r hence rotation will leave it invariant. The kinetic term can be expanded into:

$$\dot{\vec{r}}^2 = \dot{\vec{r}} \cdot \dot{\vec{r}} \to (\dot{\vec{r}} + \vec{\alpha} \times \dot{\vec{r}}) \cdot (\dot{\vec{r}} + \vec{\alpha} \times \dot{\vec{r}}) = \dot{\vec{r}}^2 + (\vec{\alpha} \times \dot{\vec{r}}) \cdot \dot{\vec{r}} + \dot{\vec{r}} \cdot (\vec{\alpha} \times \dot{\vec{r}}) + (\vec{\alpha} \times \dot{\vec{r}}) \cdot (\vec{\alpha} \times \dot{\vec{r}}) \tag{1.8}$$

clearly the 2nd and 3rd term vanishes since they are dot products of orthogonal vectors. The last term also vanish since it involves $O(\alpha^2)$. The action is then invariant and without boundary term, thus we must have $K = 0$.

Example 1.2 (Time translation)**.** Again we start from the action of a point particle under a central force:

$$S[\vec{r}(t)] = \int dt \left(\frac{m}{2}\dot{\vec{r}}^2(t) - V(r)\right) \tag{1.9}$$

but with a different symmetry operation:

$$\delta_s \vec{r}(t) = -\epsilon \dot{\vec{r}}(t) \tag{1.10}$$

with ϵ a small constant. This is essentially a time translation in the following sense:

$$r(t - \Delta t) = r(t) - \dot{r}(t)\Delta t + O(\Delta t^2) \quad \Rightarrow \quad r(t - \Delta t) - r(t) \approx -\dot{r}(t) \tag{1.11}$$

The kinetic term of Lagrangian then changes by:

$$(\dot{\vec{r}} + \delta_s \dot{\vec{r}}) \cdot (\dot{\vec{r}} + \delta_s \dot{\vec{r}}) - \dot{\vec{r}}^2 = 2\dot{\vec{r}} \cdot \delta_s \dot{\vec{r}} = -2\epsilon \dot{\vec{r}} \cdot \ddot{\vec{r}} \tag{1.12}$$

and $V(r)$ changes by $\nabla V(r) \cdot \dot{\vec{r}}$. So the change in action is

$$\begin{aligned} \delta S[\vec{r}, \delta_s \vec{r}] &= \int dt \, \epsilon \left(-m\dot{\vec{r}} \cdot \ddot{\vec{r}} + \nabla V \cdot \dot{\vec{r}}\right) \\ &= \int dt \frac{d}{dt} \left(-\epsilon \frac{m}{2} \dot{\vec{r}} + \epsilon V(\vec{r})\right) \end{aligned} \tag{1.13}$$

so the boundary term is

$$K = -\epsilon \frac{m}{2} \dot{\vec{r}} + \epsilon V(\vec{r}) \tag{1.14}$$

1.2 On-shell variations

The on-shell variation is another type of variation compared to the previous variation according to a defined operation. For symmetries, the variations $\delta_s q^i(t)$ are constrained to satisfy an equation, while the "fields" $q^i(t)$ are totally arbitrary. **For on-shell variations, the fields $q_i(t)$ are constrained to satisfy their Euler-Lagrange equations while the variations $\delta q^i(t)$ are arbitrary**.

Let $\delta q^i(t)$ be an arbitrary infinitesimal deformation of the variable $q^i(t)$. Then, for an action of the form $S[q] = \int dt L(q, \dot{q})$ the variation $\delta S[q] = S[q + \delta q] - S[q]$ can be written as

$$\begin{aligned}\delta S[q^i, \delta q^i] &= \int dt \left(\frac{\partial L}{\partial q^i} \delta q^i + \frac{\partial L}{\partial \dot{q}^i} \delta q^i \right) \\ &= \int dt \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} \right) \delta q^i + \int dt \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \delta q^i \right)\end{aligned} \tag{1.15}$$

If $q^i(t) = \bar{q}^i(t)$ satisfies its Euler-Lagrange equations, the bulk contribution vanishes and the variation is a total derivative:

$$\delta S[\bar{q}^i(t), \delta q^i(t)] = S[\bar{q}^i(t) + \delta q^i(t)] - S[\bar{q}^i(t)] = \int dt \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \delta q^i \right) \tag{1.16}$$

1.3 Noether's First theorem

The combination of a symmetry with an on-shell variation gives rise to Noether theorem. Recall that a symmetry is defined by:

$$\delta S[q^i(t), \delta_s q^i(t)] = \int dt \frac{dK}{dt} \tag{1.17}$$

and an on-shell variation is defined by

$$\delta S[\bar{q}^i(t), \delta q^t_i] = \int dt \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \delta q^i \right) \tag{1.18}$$

It is important to point out that, the two variations are from different origins: Eq.(1.17) says $q^i(t)$ is arbitrary but $\delta_s q^i(t)$ is constrained to satisfy some action symmetry, Eq.(1.18) says the field $\bar{q}^i(t)$ is fixed to satisfy Euler-Larange eqaution but $\delta q^i(t)$ is arbitrary.

Since Eq.(1.17) has no constraint on field $q^i(t)$, we have the freedom to chose $q^i(t) = \bar{q}^i(t)$, and get

$$\delta S[\bar{q}^i(t), \delta_s q^i(t)] = \int dt \frac{dK}{dt} \tag{1.19}$$

and since Eq.(1.18) has no constraint on variation $\delta q^i(t)$, we have the freedom to chose $\delta q^i(t) = \delta_s q^i(t)$, and get

$$\delta S[\bar{q}^i(t), \delta_s q^t_i] = \int dt \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \delta q^i \right) \tag{1.20}$$

Subtracting one from the other, we have

$$\frac{d}{dt} \left(K - \frac{\partial L}{\partial \dot{q}^i} \delta_s q^i \right) = 0 \tag{1.21}$$

Hence if we define

$$Q \equiv K - \frac{\partial L}{\partial \dot{q}^i} \delta_s q^i \tag{1.22}$$

then it must be a conserved quantity since $dQ/dt = 0$. This is Noether's First theorem: Given a variation of symmetry $\delta_s q^i(t)$, $Q \equiv K - \frac{\partial L}{\partial \dot{q}^i} \delta_s q^i$ is conserved.

Example 1.3 (Conservation of angular momentum)**.** As discussed in previous section, the boundary term of rotational symmetry is $K = 0$. Hence the conserved change w.r.t to $\vec{\alpha}$ is

$$Q_\alpha = -\frac{\partial L}{\partial \dot{q}^i}\delta_s q^i = -m\dot{\vec{r}} \cdot \delta_s \vec{r} = -m\dot{\vec{r}} \cdot (\vec{\alpha} \times \vec{r}) = -\vec{\alpha} \times \left(m\vec{r} \times \dot{\vec{r}}\right) \tag{1.23}$$

note here $q^i = x, y, z$. Since $\vec{\alpha}$ is an arbitrary c-number, angular momentum $\vec{L} = m\vec{r} \times \dot{\vec{r}}$ must be conserved.

Example 1.4 (Conservation of energy)**.** Recall from previous section that the boundary term from time translation reads

$$K = -\epsilon\frac{m}{2}\dot{\vec{r}}^2 + \epsilon V(\vec{r}) \tag{1.24}$$

and from Lagrangian $L = (m/2)\dot{r}^2 - V(r)$ we have

$$\frac{\partial L}{\partial \dot{q}^i}\delta_s q^i = m\dot{\vec{r}} \cdot (-\epsilon\dot{\vec{r}}) = -\epsilon m\dot{\vec{r}}^2 \tag{1.25}$$

so we have

$$Q = \frac{m}{2}\dot{\vec{r}}^2 + V(\vec{r}) \equiv E \tag{1.26}$$

Example 1.5 (The *conformal* particle)**.** In this example we will see how one can solve the equation of motion without even having to write them down. Just by looking at the symmetries and making use of Noether's theorem, one can completely integrate the dynamics.

Consider a particle of mass m under the influence of an inverse quadratic potential:

$$S[x] = \int dt \left(\frac{1}{2}m\dot{x}^2 - \frac{\alpha}{x^2}\right) \tag{1.27}$$

This action satisfies the *Weyl* symmetry:

$$t \to t' = \lambda t \tag{1.28}$$

$$x \to x'(t') = \sqrt{\lambda}x(t) \tag{1.29}$$

for constant λ. Now let us check this is an action symmetry. Under this transormation we have

$$\dot{x} = \frac{dx}{dt} \to \dot{x}' = \frac{d\sqrt{\lambda}x}{d(\lambda t)} = \frac{1}{\sqrt{\lambda}}\dot{x} \tag{1.30}$$

then it's trivial to see the $S[x'] = S[x]$. Note here we only verified the existence of such symmetry, but we don't yet know whether $K = 0$ or $K \neq 0$, since we haven't yet write down how S changes under infinitesimal variation. To do this, let $\lambda = 1 + \epsilon$ and expand the transformation into:

$$x(t) \to x'(t') = x'((1+\epsilon)t) = \sqrt{1+\epsilon}\, x(t) \approx \left(1 + \frac{\epsilon}{2}\right) x(t) \tag{1.31}$$

and the LHS can be expand to linear order as

$$x'((1+\epsilon)t) = x'(t) + \frac{dx'}{dt}\epsilon t \approx x'(t) + \dot{x}(t)\epsilon t \tag{1.32}$$

where we have used a result that follows Eq.(1.30) and ignored $O(\epsilon^2)$:

$$\frac{dx'}{dt} = \frac{dx'}{dt'}(1+\epsilon) = \sqrt{1+\epsilon}\frac{dx}{dt} \approx \left(1 + \frac{\epsilon}{2}\right)\dot{x} \tag{1.33}$$

Hence by comparing Eq.(1.32) and Eq.(1.31) the variation can be written as

$$\delta_s x(t) = x'(t) - x(t) = -\epsilon t\dot{x} + \frac{\epsilon}{2}x \tag{1.34}$$

In order to apply this to action, we need first calculate how each term of integrand changes under such variation. The $\dot{x}^2$ term becomes

$$\dot{x}^2 \to \left(\frac{d}{dt}\left(x+\delta_s x\right)\right)^2 = \left[\dot{x} + \frac{d}{dt}(-\epsilon t\dot{x} + \frac{\epsilon}{2}x)\right]^2 \approx \dot{x}^2 - \epsilon\left(t\dot{x}\ddot{x} + \frac{1}{2}\dot{x}^2\right) \tag{1.35}$$

so that

$$\delta(\dot{x}^2) = -\epsilon\left(t\dot{x}\ddot{x} + \frac{1}{2}\dot{x}^2\right) \tag{1.36}$$

By the same token, the $\delta(1/x^2)$ term gives:

$$\begin{aligned}
\delta\left(\frac{1}{x^2}\right) &= \frac{1}{(x+\delta_s x)^2} - \frac{1}{x^2} \approx \frac{1}{x^2 + 2x\,\delta_s x} - \frac{1}{x^2} \\
&= \frac{x^2 - x^2 - 2x\,\delta_s x}{x^2(x^2 + 2x\,\delta_s x)} = \frac{-2\delta_s x}{x^3 + 2x^2\delta_s x} = \frac{1}{x^3}\frac{-2\delta_s x}{1 + (2\delta_s x/x)} \\
&= \frac{-2\delta_s x}{x^3}\left(1 - \frac{2\delta_s x}{x} + O(\delta_s x^2)\right) \\
&\approx -\frac{2\delta_s x}{x^3} = \epsilon\frac{2t\dot{x} - x}{x^3}
\end{aligned} \tag{1.37}$$

it is clear now that δ can indeed be treated like a usual differential d. Now we are ready to evalulate the boundary term:

$$\begin{aligned}
\delta S[x] &= \int dt\left(\frac{m}{2}\delta\left(\dot{x}^2\right) - \alpha\delta\left(\frac{1}{x^2}\right)\right) \\
&= \epsilon\int dt\left[-m\left(\frac{1}{2}\dot{x}^2 + t\dot{x}\ddot{x}\right) + \alpha\frac{x - 2t\dot{x}}{x^3}\right] \\
&= \epsilon\int dt\frac{d}{dt}\left[-m\left(\frac{t\dot{x}^2}{2}\right) + \frac{\alpha t}{x^2}\right]
\end{aligned} \tag{1.38}$$

thus we define:

$$K \equiv -\frac{m}{2}t\dot{x}^2 + \frac{\alpha t}{x^2} \tag{1.39}$$

Then it is readily to derive Q

2 Noether's theorem in Hamiltonian mechanics

3 Symmetries act on fields

3.1 Spacetime translation

Spacetime translations $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu$ are understood as a transformation of the field as follows: Given $\phi(x)$, we can build a new field $\phi'(x')$ supported on $x' = x + \epsilon$. The new and the old are necessarily related by

$$\phi(x) \to \phi'(x+\epsilon) = \phi(x) \tag{3.1}$$

This is generically true for all x under such translation. Substituting $x-\epsilon$ for x gives us the desired form (since we are interested in how the new field looks like in the original fixed coordinate):

$$\phi'(x) = \phi(x-\epsilon) \tag{3.2}$$

Then by expanding to 1st order of ϵ we have

$$\phi'(x) = \phi(x-\epsilon) \simeq \phi(x) - \epsilon^\mu \partial_\mu \phi(x) \tag{3.3}$$

Therefore the variation of the field by spacetime translation $x' = x + \epsilon$ is

$$\delta\phi(x) = -\epsilon^\mu \partial_\mu \phi(x) \tag{3.4}$$

For the simplest scalar field

$$I[\phi(x)] = \frac{1}{2}\int d^4x\, \partial_\mu\phi\partial^\mu\phi \tag{3.5}$$

its variation under spacetime translation is

$$\begin{aligned}
\delta I[\phi] &= \frac{1}{2}\int d^4x \partial_\mu[\phi - \epsilon^\nu\partial_\nu\phi]\partial^\mu[\phi - \epsilon^\nu\partial_\nu\phi] - \frac{1}{2}\int d^4x\, \partial_\mu\phi\partial^\mu\phi \\
&= \frac{1}{2}\int d^4x\, \partial_\mu\phi\partial^\mu(-\epsilon^\nu\partial_\nu\phi) + \frac{1}{2}\int d^4x\, \partial_\mu(-\epsilon^\nu\partial_\nu\phi)\partial^\mu\phi \\
&= \int d^4x\, \partial_\mu\phi\partial^\mu(-\epsilon^\nu\partial_\nu\phi) \\
&= -\frac{1}{2}\int d^4x\, \partial_\nu(\epsilon^\nu\partial_\mu\phi\partial^\mu\phi)
\end{aligned} \tag{3.6}$$

where we ignored $O(\epsilon^2)$. So $\delta I[\phi]$ is only a boudary term

Note that any potential term $U(\phi)$ that does not explicitly depend on x does not spoil the aforesaid symmetry, because:

$$\delta U = U(\phi + \delta\phi) - U(\phi) = \frac{dU(\phi)}{d\phi}\delta\phi = -U'(\phi)\epsilon^\mu\partial_\mu\phi = -\partial_\mu(\epsilon^\mu U) \tag{3.7}$$

which again is only a boundary term.

3.2 Lorentzian boost

An arbitrary Lorentz transformation is

$$x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu \tag{3.8}$$

Again we can define a new field $\phi'(x')$ which is related by

$$\phi'(\Lambda^\mu_\nu x^\nu) = \phi(x) \tag{3.9}$$

Now we substitute $x = \Lambda^{-1}x$ and we have

$$\phi'(x) = \phi(\Lambda^{-1}x) \tag{3.10}$$

By the chain rule, the derivative of the field $\partial_\mu\phi(x)$ is transformed as:

$$\partial_\mu\phi(x) \to \partial_\mu\phi'(x) = \partial_\mu\phi(\Lambda^{-1}x) = \left(\frac{\partial(\Lambda^{-1})^\nu_\rho x^\rho}{\partial x^\mu}\right)(\partial_\nu\phi)(\Lambda^{-1}x) = (\Lambda^{-1})^\nu_\mu(\partial_\nu\phi)(\Lambda^{-1}x) \tag{3.11}$$

note that it is only the field that is considered to have changed during the boost, thus ∂_μ remains the same. The second order derivative is

$$\begin{aligned}\partial_\mu \partial^\mu &= \left[(\Lambda^{-1})^\nu_\mu \partial_\nu\right]\left[(\Lambda^{-1})^\sigma_\lambda \partial_\sigma\right] g^{\lambda\mu} \\ &= \left[(\Lambda^{-1})^\nu_\mu (\Lambda^{-1})^\sigma_\lambda g^{\lambda\mu}\right] \partial_\nu \partial_\sigma = g^{\sigma\nu}\partial_\nu\partial_\sigma = \partial_\sigma \partial^\sigma\end{aligned} \tag{3.12}$$

where we used the identity:

$$(\Lambda^{-1})^\rho_\mu (\Lambda^{-1})^\sigma_\nu g^{\mu\nu} = g^{\rho\sigma} \tag{3.13}$$

so the $\partial_\mu \partial^\mu$ is Lorentzian invariant.

Example 3.1. Show that

$$S = (\partial_t \psi)(\partial_t \phi) \tag{3.14}$$

is Lorentzian invariant.

Proof. According to Eq.3.11 we have

$$\begin{aligned}S \to S' &= \partial_0 \psi(\Lambda^{-1}x) \cdot \partial_0 \phi(\Lambda^{-1}x) \\ &= (\Lambda^{-1})^\nu_0 (\Lambda^{-1})^\mu_0 (\partial_\nu \psi)(\partial_\mu \phi)(\Lambda^{-1}x) = (\partial_0\psi)(\partial_0\phi)(\Lambda^{-1}x)\end{aligned} \tag{3.15}$$

since $\Lambda^\nu_0 = \Lambda^\nu_0 \delta_{0,\nu}$ □

conserved quantity

3.3 Conserved current and conserved charge

For a d-dimensional theory the action reads in general

$$I[\phi(x)] = \int d^dx\ \mathcal{L}(\phi, \partial_\mu \phi) \tag{3.16}$$

which corresponds to the Euler-Lagrange equation:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{3.17}$$

The on-shell variation is computed as

$$\begin{aligned}\delta I[\bar{\phi}, \delta\phi] &= \int d^dx \left(\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu\phi)\right) \\ &= \int d^dx \left(\left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\right]\delta\phi\right) + \int d^dx\ \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi\right) \\ &= \int d^dx\ \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi\right) = \delta I[\bar{\phi}, \delta_s\phi]\end{aligned} \tag{3.18}$$

and the symmetry variation is

$$\delta I[\phi, \delta_s\phi] = \int d^dx\ \partial_\mu K^\mu = \delta I[\bar{\phi}, \delta_s\phi] \tag{3.19}$$

by the two equations we have the conserved current equation:

$$\partial_\mu J^\mu = 0,\ \ J^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta_s\phi(x) - K^\mu \tag{3.20}$$

To the the conserved charge explicitly, we split the spacial and temporal part of conserved current:

$$\partial_t J^0 = -\nabla \cdot \vec{J} \tag{3.21}$$

Integrating both sides of this equation and using the divergence theorem

$$\int_V d^{d-1}x \ \partial_t J^0 = -\int d^{d-1}x \ \nabla \cdot \vec{J} = -\int_{\partial V} \vec{J} \cdot \vec{S} \tag{3.22}$$

If the container V is large enough (and assuming field configurations such that $\vec{J}$ drops to zero faster than the growth of the surface area) the last integral vanishes, yielding the conserved charge

$$\frac{dQ}{dt} = 0, \ Q = \int_V d^{d-1}x \ J^0(x) \tag{3.23}$$

Actually, this widespread phrase that "fields fall off sufficiently rapidly at infinity" will turn out to be false in gauge theories.

3.4 $U(1)$ symmetry

Consider the complex scalar field theory:

$$I[\psi(x), \psi^\dagger(x)] = \int d^4x \left[(\partial_\mu \psi)^\dagger \partial^\mu \psi - V(\psi^\dagger \psi) \right] \tag{3.24}$$

Assuming $V(\psi^\dagger\psi) = m\psi^\dagger\psi$, the equation of motions are then given by the Klein-Gordon equation

$$(\partial^2 + m^2)\psi = 0, \ (\partial^2 + m^2)\psi^\dagger = 0 \tag{3.25}$$

Obviouly the action is invariant under $U(1)$ transformation:

$$\psi \to \psi' = e^{-i\alpha}\psi, \quad \psi^\dagger \to \psi'^\dagger = e^{i\alpha}\psi^\dagger \tag{3.26}$$

where α is a global constant, making the transformation a global $U(1)$ operation. The infinitesmal generator of the $U(1)$ transformation is

$$\psi \to \psi' = e^{-i\alpha}\psi \approx \psi - i\alpha\psi, \quad \psi^\dagger \approx \psi^\dagger + i\alpha\psi^\dagger \tag{3.27}$$

The on-shell variation gives

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\delta_s\psi = (\partial^\mu\psi)^\dagger\delta_s\psi = -i\alpha\psi\partial^\mu\psi^\dagger \tag{3.28}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\dagger)}\delta_s\psi^\dagger = (\partial^\mu\psi)\delta_s\psi^\dagger = i\alpha\psi^\dagger\partial^\mu\psi \tag{3.29}$$

The symmetry variation gives zero because in the transformed field the Lagrangian takes the same form:

$$V[(\psi^\dagger + i\alpha\psi^\dagger)(\psi - i\alpha\psi)] = V[\psi^\dagger\psi + i\alpha\psi^\dagger\psi - i\alpha\psi\psi^\dagger + O(\alpha^2)] \approx V(\psi^\dagger\psi) \tag{3.30}$$

$$\partial^\mu(\psi - i\alpha\psi)\partial_\mu(\psi^\dagger + i\alpha\psi^\dagger) = \partial^\mu\psi\partial_\mu\psi^\dagger - i\alpha\partial^\mu\psi\partial_\mu\psi + i\alpha\partial^\mu\psi\partial_\mu\psi^\dagger + O(\alpha^2) \approx \partial^\mu\psi\partial_\mu\psi^\dagger \tag{3.31}$$

so the total derivative from symmetry variation is $K^\mu = 0$, and the Noether current is determined fully by the on-shell variation:

$$j^\mu = i(\psi\partial^\mu\psi^\dagger - \psi^\dagger\partial^\mu\psi) \tag{3.32}$$

It is simple to check the current satisfies continuity equation by using the eoms:

$$\partial_\mu j^\mu = i(\psi^\dagger \partial^2 \psi - \psi^2 \psi^\dagger) = i(-\psi^\dagger m^2 \psi + \psi m^2 \psi^\dagger) = 0 \tag{3.33}$$

and the conserved charge is written as

$$Q = \int d^3x\ j^0 = i \int d^3x\ (\psi \dot{\psi}^\dagger - \psi^\dagger \dot{\psi}) \tag{3.34}$$

4 Appendix

4.1 Infinitesimal rotation

Consider a infinitesimally small rotation $R = I + A$, the orthogonality demands

$$R^T R = (I + A^T)(I + A) \approx I + A^T + A = I$$

This requires

$$A^T = -A$$

namely, that A must be antisymmetric. In 2D, there is only one class of antisymmetric rotation matrix:

$$A = \epsilon \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \equiv \epsilon \mathcal{J}$$

where ϵ is a small real number and $\mathcal{J}$ is termed the generator of $SU(2)$. Hence for a small angle, rotation matrix can be written as

$$R = I + \epsilon \mathcal{J} = \begin{pmatrix} 1 & \epsilon \\ -\epsilon & 1 \end{pmatrix} + O(\epsilon^2)$$

Under such rotation we have

$$\begin{pmatrix} x' \\ y' \end{pmatrix} \approx \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \epsilon y \\ -\epsilon x \end{pmatrix}$$

This immediately gives another useful form of infinitesimal rotation:

$$\mathbf{r}' = \mathbf{r} + \epsilon \times \mathbf{r}$$

where ϵ points into the $\hat{z}$ direction.