Non-Abelian behavior of Kitaev spin liquid

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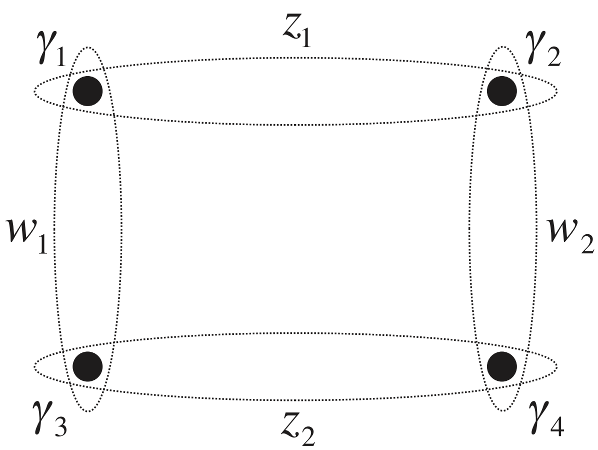
In Kitaev's honeycomb model, it is known that vortex (flux) excitation has non-Abelian statistics which makes the model one of the candidates for topological quantum computation. In this draft, I demonstrate that it is not the vortex excitation *par se* that are responsible for the non-Abelianess, but the majorana particles bounded to these vortices which give the non-Abelian behavior of vortices. Indeed, the chiral spin liquid by gapping out the complex fermion modes in Kitaev model has a non-zero Chern number, indicating the existence of half-quantized chiral majorana modes living on magnetic defects or Z_2 flux excitation. Two such fluxes distant apart then become the non-local bearer of a complex fermion mode by having one majorana attached to each flux. The most striking behavior in Kitaev model is that the majorana modes carried by these fluxes exhibit non-Abelian statistics which can be exploited to make quantum gates in the fermion parity subspace – since complex fermions here effectively form $p_x + ip_y$ superconductor. In the coming text I review how to perceive the non-Abelian behavior by manipulating localized majoranas. More details can be found in the friendly reference Pachos (2012).

1 Fusion

Let us think of four localized majorana $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ that lives at four vortices distant apart from each other. Alone, an isolated majorana is not physical, however, we can group them into pairs such that two majoranas are combined into fermionic modes. There are two ways to group these four majoranas, as shown in Fig. 1. Let us then define these fermion modes as



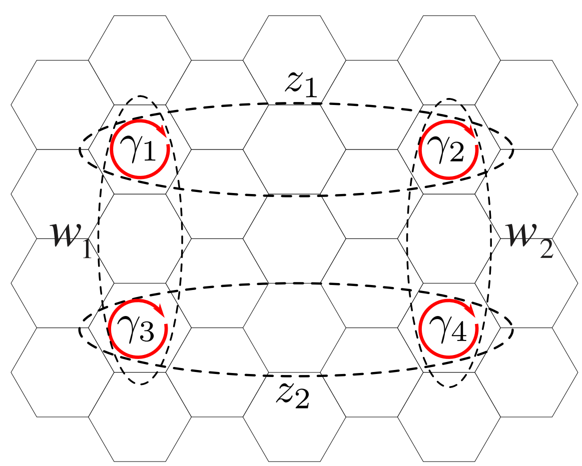


Figure 1: Left: Four majoranas can make different fermionic modes. Right: microscopic realization in Kitaev model

$$z_1 = \frac{1}{2}(\gamma_1 + i\gamma_2), \quad z_2 = \frac{1}{2}(\gamma_3 + i\gamma_4), \quad w_1 = \frac{1}{2}(\gamma_1 + i\gamma_3), \quad w_2 = \frac{1}{2}(\gamma_2 + i\gamma_4) \tag{1}$$

which satisfies the anti-commutation relation:

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}, \quad \{z_i, z_j^\dagger\} = 1, \quad \{w_i, w_i^\dagger\} = 1 \tag{2}$$

However, it is interesting and important to note that the two fermionic modes z and w are not independent, such that $\{z_i, w_i\} \neq 0$, since the the two modes share a common majorana. It is simple to check the following

$$\{z_1, w_1\} = \{\frac{1}{2}\gamma_1, \frac{1}{2}\gamma_1\} = \frac{1}{2}, \quad \{z_2, w_2\} = \{\frac{i}{2}\gamma_4, \frac{i}{2}\gamma_4\} = -\frac{1}{2} \tag{3}$$

We focus on the subspace of even fermion parity, i.e. fermionic modes fuse to the vacuum. This is the case, for example, in a SC where two fermions form a Cooper pair. We consider the fermion occupation basis:

$$|ij\rangle_z \equiv |i\rangle_z \otimes |j\rangle_z\,, \quad |ij\rangle_w \equiv |i\rangle_w \otimes |j\rangle_w \quad i,j \in \{0,1\} \tag{4}$$

As w and z modes are not independent, the modes in z basis can be written as a linear combination of modes in w basis, for example

$$|00\rangle_z = \alpha\,|00\rangle_w + \beta\,|11\rangle_w \tag{5}$$

and so on. This allows us to connect occupation modes in different basis by number operators. To appreciate this point, we first note that the number operators of these modes can be viewed as projectors: $z_i^\dagger z_i$ and $w_i^\dagger w_i$ projects out the zero population states, while $z_i z_i^\dagger$ and $w_i w_i^\dagger$ projects out the populated states. To be clear, it is readily to check the following:

$$z_i^\dagger z_i\,|00\rangle_z = 0, \quad z_i z_i^\dagger\,|00\rangle_z = (1 - z_i^\dagger z_i)\,|00\rangle_z = |00\rangle_z \tag{6}$$

$$z_i^\dagger z_i\,|11\rangle_z = |11\rangle\,, \quad z_i z_i^\dagger\,|11\rangle_z = (1 - z_i^\dagger z_i)\,|11\rangle_z = 0 \tag{7}$$

and the projectors made of w and $w^\dagger$ can be constructed by the same token:

$$w_i^\dagger w_i\,|00\rangle_w = 0, \quad w_i w_i^\dagger\,|00\rangle_w = (1 - w_i^\dagger w_i)\,|00\rangle_w = |00\rangle_w \tag{8}$$

$$w_i^\dagger w_i\,|11\rangle_w = |11\rangle\,, \quad w_i w_i^\dagger\,|11\rangle_w = (1 - w_i^\dagger w_i)\,|11\rangle_w = 0 \tag{9}$$

Therefore, as an example, it is now readily to see that the operator $w_1^\dagger w_1$ projects the state $|00\rangle_z$ and $|11\rangle_z$ onto $|11\rangle_w$, because

$$w_1^\dagger w_1\,|00\rangle_z = w_1^\dagger w_1(\alpha\,|00\rangle_w + \beta\,|11\rangle_w) \propto |11\rangle_w \tag{10}$$

$$w_1^\dagger w_1\,|11\rangle_z = w_1^\dagger w_1(\alpha'\,|00\rangle_w + \beta'\,|11\rangle_w) \propto |11\rangle_w \tag{11}$$

and similarly, we have the projector $w_1 w_1^\dagger$ which selects out $|00\rangle_w$:

$$w_1 w_1^\dagger\,|00\rangle_z = w_1 w_1^\dagger\,|11\rangle_z = |00\rangle_w \tag{12}$$

Next, we would like to figure out the relation between them. In other words, we need to find the following matrix elements F_{ij}:

$$|11\rangle_w = F_{01}\,|00\rangle_z\,, \quad |11\rangle_w = F_{11}\,|11\rangle_z\,, \quad |00\rangle_w = F_{00}\,|00\rangle_z\,, \quad |00\rangle_w = F_{10}\,|11\rangle_z \tag{13}$$

We start from the construction of F_{10}. Let us look at the state given by $(2w_1^\dagger w_1 - 1)\left|00\right\rangle_z$ first. What is the occupation in the z basis? It is straightforward to calculate the following:

$$z_1^\dagger z_1(2w_1^\dagger w_1 - 1)\left|00\right\rangle_z = (2w_1^\dagger w_1 - 1)(1 - 2z_1^\dagger z_1)\left|00\right\rangle_z = (2w_1^\dagger w_1 - 1)\left|00\right\rangle_z \tag{14}$$

in other words, $(2w_1^\dagger w_1 - 1)\left|00\right\rangle_z$ is a non-trivial eigen state of occupation number $z_1^\dagger z_1$. Hence, in the even parity subspace we have but one choice:

$$\left|11\right\rangle_z = (2w_1^\dagger w_1 - 1)\left|00\right\rangle_z \tag{15}$$

Note that the operator $w_1^\dagger w_1$ projects the state $\left|00\right\rangle_z$ into $\left|11\right\rangle_w$, we can rewrite the above relation as:

$$\left|11\right\rangle_w \propto 2w_1^\dagger w_1 = \left|11\right\rangle_z + \left|00\right\rangle_z \tag{16}$$

After nomalization we have

$$\boxed{\left|11\right\rangle_w = \sqrt{2}w_1^\dagger w_1\left|00\right\rangle_z = \frac{1}{\sqrt{2}}(\left|00\right\rangle_z + \left|11\right\rangle_z)} \tag{17}$$

We can now apply the anti-commutation $\{w_i, w_i^\dagger\} = 1$ to Eq. 17, and arrive at

$$\sqrt{2}(1 - w_1 w_1^\dagger)\left|00\right\rangle_z = \frac{1}{\sqrt{2}}(\left|00\right\rangle_z + \left|11\right\rangle_z) \;\Rightarrow\; \sqrt{2}w_1 w_1^\dagger\left|00\right\rangle_z = \frac{1}{\sqrt{2}}(\left|00\right\rangle_z - \left|11\right\rangle_z) \tag{18}$$

since $w_1 w_1^\dagger$ projects $\left|00\right\rangle_z$ onto $\left|00\right\rangle_w$, we have

$$\boxed{\left|00\right\rangle_w = \sqrt{2}w_1 w_1^\dagger\left|00\right\rangle_z = \frac{1}{\sqrt{2}}(\left|00\right\rangle_z - \left|11\right\rangle_z)} \tag{19}$$

Hence the two set of fermionic modes are related by the matrix

$$\begin{pmatrix}\left|00\right\rangle_w \\ \left|11\right\rangle_w\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\left|00\right\rangle_z \\ \left|11\right\rangle_z\end{pmatrix} \tag{20}$$

up to an exchange of basis vectors. This is the fusion matrix F

$$\boxed{F = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1 \\ 1 & 1\end{pmatrix}} \tag{21}$$

2 Braiding

Let us now turn to the braiding properties of majoranas γ_i and γ_j, shown schematically in Fig. 2(a). The unitary operator that can act on the majorana state is given by

$$U = \frac{e^{i\theta}}{\sqrt{2}}(1 + \gamma_i\gamma_j), \quad U' = \frac{e^{i\phi}}{\sqrt{2}}(\gamma_i + \gamma_j) \tag{22}$$

It is easy to check that these unitary transformations exchange two majoranas:

$$U\gamma_i U^\dagger = \gamma_j, \quad U'\gamma_i U'^\dagger = \gamma_j \tag{23}$$

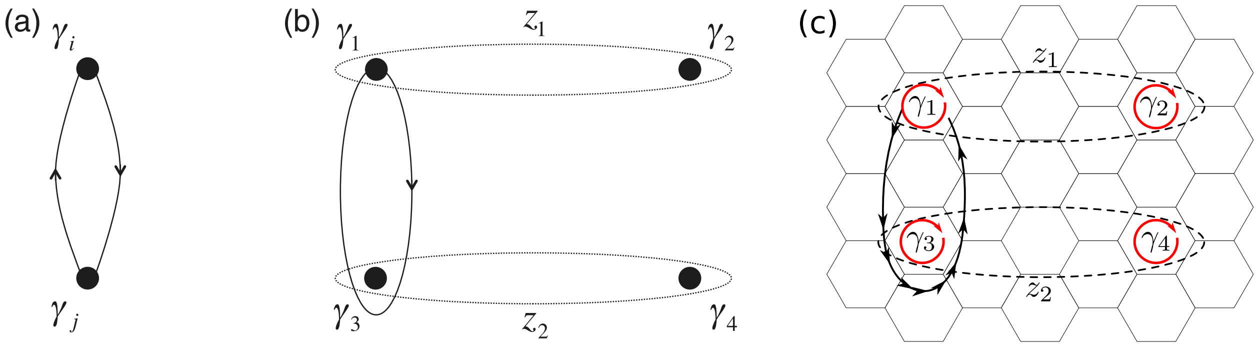


Figure 2: (a) The exchange of two majoranas, γ_i and γ_j. (b) Four majoranas that give rise to two fermionic modes z_1 and z_2. The occupation states of these two modes $|ij\rangle_z$ can be changed from one into another by, for example, braiding γ_1 around γ_3 by exchanging twice via U^2. (c) the corresponding process illustrated in Kitaev's honeycomb lattice.

The U' operator is Abelian, since $U'^2 = e^{i2\phi}$ which is just a phase factor. On the other hand, the U operator is non-Abelian isnce $U^2 = e^{2i\theta}\gamma_i\gamma_j$ is still another operation.

Now we would like to know how does braiding change the fermionic states. Consider the operation $U^2 = e^{2i\theta}\gamma_1\gamma_3$ which exchanges γ_1 and γ_3 twice, making an effective 2π braiding between these majoranas. Since we are interested in how braiding affect the fermionic space, we convert the majoranas back to fermion occupation states:

$$\gamma_1 = ..., \quad \gamma_3 = ... \tag{24}$$

giving us

$$U^2 = e^{2i\theta}\gamma_1\gamma_3 = e^{2i\theta}(z_1 z_2 + z_1 z_2^\dagger + z_1^\dagger z_2 + z_1^\dagger z_2^\dagger) \tag{25}$$

as is shown in Fig. 2(b). Hence U^2 is a gate that switch between states:

$$U^2 |00\rangle_z = |11\rangle_z, \quad U^2 |11\rangle_z = |00\rangle_z \tag{26}$$

$$U^2 |01\rangle_z = |10\rangle_z, \quad U^2 |10\rangle_z = |01\rangle_z \tag{27}$$

therefore, U^2 is an exchange matrix upto a phase factor:

$$U^2 \sim \sigma^x \tag{28}$$

which is the expected non-Abelian action of the Ising anyon. Next let us figure out the phase factor θ. This can be done by considering a sequence of braiding between three majoranas, as shown in Fig. 3. In this composite braiding, initially γ_1 braids around γ_2 and γ_3 in clockwise direction, denoted by operation B_1. Then γ_2 and γ_3 fuse into a fermion $\alpha = (\gamma_2 + i\gamma_3)/2$, and γ_1 braids around the fused α in counter clockwise direction thereafter. Note that the braiding in full is trivial, since the CW braiding is undone by CCW braiding, giving

$$B_2 B_1 = 1 \tag{29}$$

however, B_2 and B_1 is a function of θ, which allows us to extract the phase factor by the above constraint. The braiding between γ_1 and the pair of γ_2, γ_3 is

$$B_1 = U_{13}^1 U_{12}^2 = e^{i4\theta}\gamma_2\gamma_3 \tag{30}$$

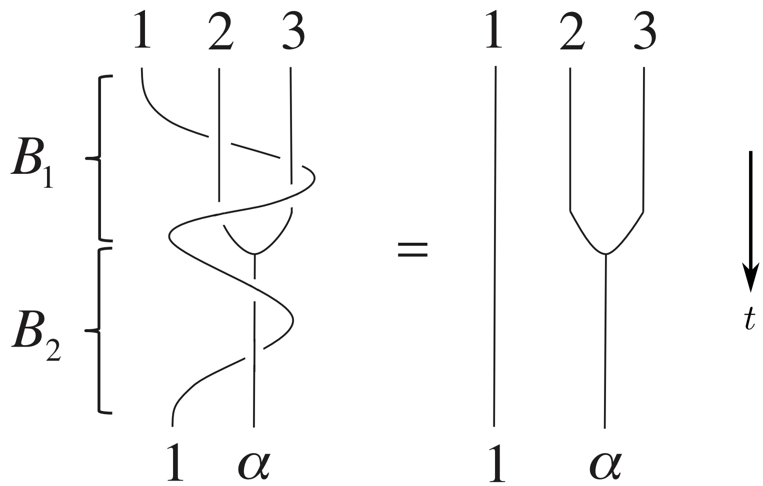


Figure 3: Braiding between three majoranas, which can be divide into two steps. B1: braiding γ_1 across γ_2 and γ_3 CW; and B2: fuse γ_3 and γ_3 into a fermion $\alpha \propto \gamma_2 + i\gamma_3$, and braid γ_1 across α CCW. The time axis goes from top to bottom.

the braiding between γ_1 and α is given by

$$B_2 = 1 - 2\alpha^\dagger\alpha = i\gamma_2\gamma_3 \tag{31}$$

which comes from the fact that the exchange between a majorana and a canonical fermion produces a phase factor i. Hence we have

$$B_2B_1 = -ie^{i4\theta} = 1 \ \Rightarrow \ \theta = \frac{\pi}{8} \tag{32}$$

and the exact braiding matrix for Fig. 2(b) is thus

$$\boxed{U^2 = e^{i\frac{\pi}{4}}\sigma^x} \tag{33}$$

References

Jiannis K. Pachos. *Introduction to Topological Quantum Computation*. Cambridge University Press, 2012. doi:10.1017/CBO9780511792908.